Predator-Prey Models

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Introduction:
History of Lotka-Volterra Model

Alfred J. Lotka originally proposed what came to be known as the Lotka-Volterra predator-prey model in 1910 in his theory of autocatalytic chemical reactions. The equation that he suggested essentially was the logistic equation derived by Pierre Francois Verhulst. Ten years later he extended the equation to include organic systems. This was achieved using an example which included a plant species and an herbivore. Later in life he used these equations more significantly in his book on biomathematics to analyze predator-prey relationships. The equations that he finalized in this text were the modern equations we have today. Vito Volterra, an Italian mathematician and physicist, also examined these equations. He used these equations when making a statistical analysis of fish catches in the Adriatic Sea. Since this was done independently of Lotka, Volterra also received credit for the existence of the equations and hence the name Lotka-Volterra was coined for the set of equations.

System of Differential Equations

A collection of multiple equations that you solve together is called a system of equations. Linear equations are simpler than non-linear equations since they graph straight lines. The easiest linear system of equations is one which contains two equations and two variables.

A nonlinear system of differential equations is a set of equations in which the unknowns are functions which appear in another function which is not a polynomial of degree one or as variables of a polynomial of degree higher than one.

A first-order system of *n* differential equations when x = $(x_1, \ldots, x_n)$, has the form:

1) $\frac{dx}{dt} = F(x)$

An *equilibrium solution* also known as a *critical point, fixed point, or steady state solution*, of the above differential equation, is a constant solution $\bar{x}$ where

$F(\bar{x})=0$

For the above mentioned differential equation with the initial condition
$x(t_0) = x_0$, an *equilibrium solution is locally stable* if for every $\epsilon>0$ there exist a $\delta>0$ so that every solution x(t) with $||x_0 - \bar{x}||<\delta$, then $||x(t)-\bar{x}||<\epsilon$, for every $t \geq t_0$.

If the equilibrium solution is not locally stable it is said to be *unstable*.

An equilibrium solution $\bar{x}$ is said to be *locally asymptotically stable* if, it is locally stable and if there exists $\gamma>0$ such that $||x_0 - \bar{x}||<\gamma \implies \lim_{t\to\infty} ||x(t) - \bar{x}|| = 0$.

A *periodic solution* of the differential system above mentioned is a nonconstant solution x(t) satisfying x(t+T) = x(t), for all t on the interval of existence and for some T > 0. The minimum value of T > 0 is called the *period* of the solution.

Let x = (x,y) and $F(x,y) = (f(x,y), g(x,y))$.
Assume the first order partial derivatives of f and g are continuous in some open set containing the equilibrium point $(\bar{x},\bar{y})$of the system

2) $\begin{cases} \frac{dx}{dt} = f(x, y) \\ \frac{dy}{dt} = g(x, y) \end{cases}$

Predator-Prey Model:
System of Differential Equations

When looking at predator prey relationships as a system of nonlinear differential equations we use the complete version of the Lotka-Volterra model. This model consists of two differential equations which are

3) $\begin{cases} \frac{dx}{dt} = f(x, y) = \text{ax} - \text{bxy} \\ \frac{dy}{dt} = g(x, y) = -\text{cy} + \text{dxy} \end{cases}$

The first equation is for the *prey: x* and the second is for the *predator: y*.
The constants a,b,c, and d > 0. Exponential growth of the prey population would result if the predator population were y=0. The predator population would decay towards 0 (extinction) if the prey population were x=0. Notice that when the predator is absent the prey grows exponentially. The following model addresses this problem as it assumes only logistic growth of the prey when the predator is absent. Essentially this model takes the carrying capacity of the ecosystem into account.

4) $\begin{cases} \frac{dx}{dt} = \text{ax} - \text{fx}^2 - \text{bxy} \\ \frac{dy}{dt} = -\text{cy} + \text{dxy} \end{cases}$

As you can see the equation for the predator is not changed. This is because we were just acknowledging that an ecosystem cannot support infinite prey if the predator is absent. If the prey is absent the predator population will still drop to 0 (extinction).

Example

Predator-Prey System of Differential Equations:

$$\begin{cases} \frac{dx}{dt} = f(x, y) = 200\,x - 4\,\text{xy} & \square \\ \frac{dy}{dt} = g(x, y) = -150\,y + 2\,\text{xy} & \square \end{cases}$$

Phase-Plane Portrait

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StreamPlot[{200 × x - 4 × x × y, -150 × y + 2 × x × y}, {x, -100, 300}, {y, -100, 200},
 Axes → True]
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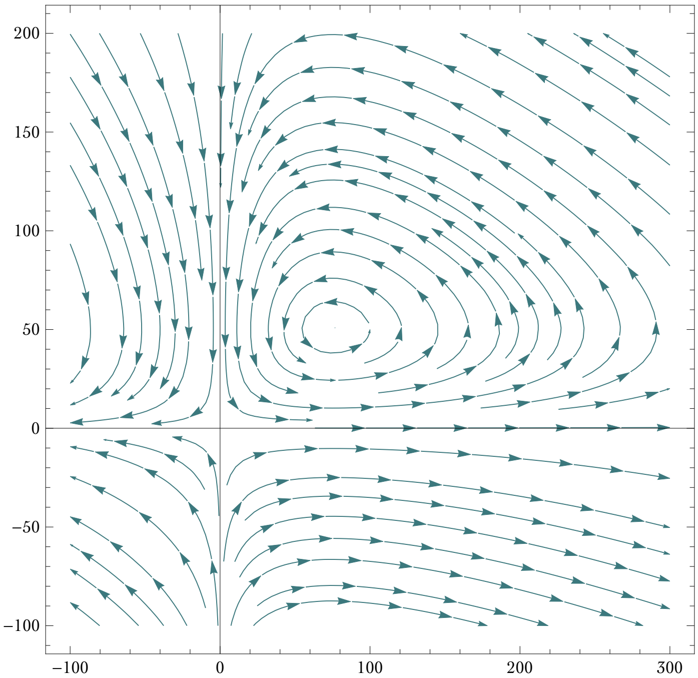

As you can see from the above phase-plane portrait the two critical points of this system are (0,0) and (75,50).

Periodic Ocsillating Predator-Prey Relationship

A periodic oscillating predator prey relationship can represented in a figure with two oscillating sinisoidal curves. These graphs are used to better show how the system (relationship) is behaving in accordance with time. This is something that cannot be seen as easily with a phase-plane portrait. The main thing to gather from the periodic oscillating relationship is that neither species population reaches the carrying capacity or 0 (extinction. Also note that there is a “lag” between the increse of the prey population and an increase in the predator population.