# HYPERBOLIC GEOMETRY

NICHOLAS SCHRODER
MAT 498
DR. VOCHITA MIHAI

## 1. Introduction

### 1.1. Euclidean Geometry.

Euclidean geometry is the study of plane and solid figures which is based on a set of axioms formulated by the greek mathematician, Euclid, in his 13 books, the Elements. Euclid was born around 300 BCE and not much is known about him, other than that he taught at Alexandria in the time of Ptolemy I Soter, a ruler of Egypt from 323-285 BCE. The axioms that Euclid formed are the basis for most of the geometry that is taught in schools today. In fact, up until the 19th century, Euclidean geometry was the only type of geometry there was. In the 19th century mathematicians found that by changing Euclids last axiom, the parallel postulate, one was able to form a whole new set of geometries. This is what led to the many different forms of geometry that we have today. These types of geometries are called non-Euclidean geometries and refer to literally any other type of geometry. The non-Euclidean geometries are spherical or elliptic and hyperbolic which will be covered later. (Artmann)

Euclid started with a set of axioms and five undefined terms: point, line, distance, half plane, angle measure, and area. Each term is introduced through the five axioms. he stated the axioms(postulates) and by using them he constructed the rest of the geometry, proving theorem and defining new terms.
The euclidian axioms are:
(1) A straight line segment can be drawn joining any two points.
(2) Any straight line segment can be extended indefinitely in a straight line.
(3) Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
(4) All right angles are congruent.
(5) If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. (This postulate is equivalent to what is known as the parallel postulate.)

1.2. **Types of geometries.** There are there types of geometries:

(1) Euclidean Geometry: the geometry where the fifth axiom is true: Through a given point outside of a given line one can construct only one parallel line with the given line.
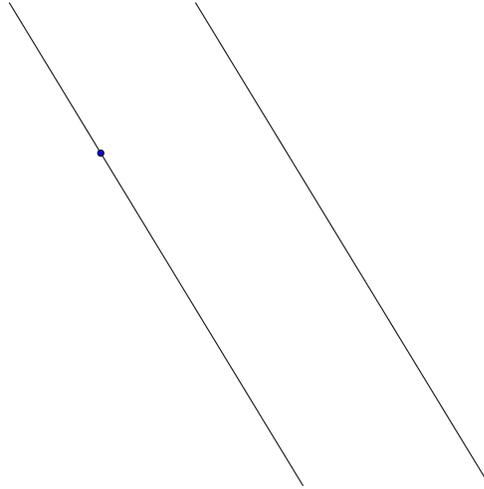


FIGURE 1. Parallel Lines

(2) Eliptic Geometry: the geometry where the fifth postulate is substitute by the following axiom: Through a given point outside of a given line one can not construct any line parallel with the given line.

For example $S^2$ satisfies the elliptic postulate: any two great circles intersect, this means that there are no parallel lines on the two-dimensional sphere.
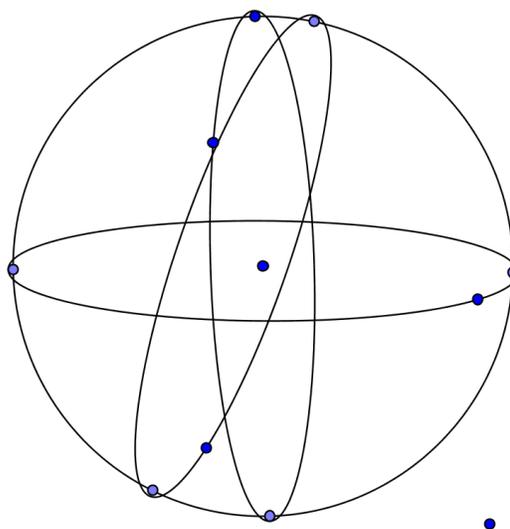


FIGURE 2. The $S^2$ Sphere

(3) Hyperbolic Geometry: the geometry where the fifth postulate is substitute with the following one: through a given point outside of a given line one can construct at least two parallel lines with the given line.
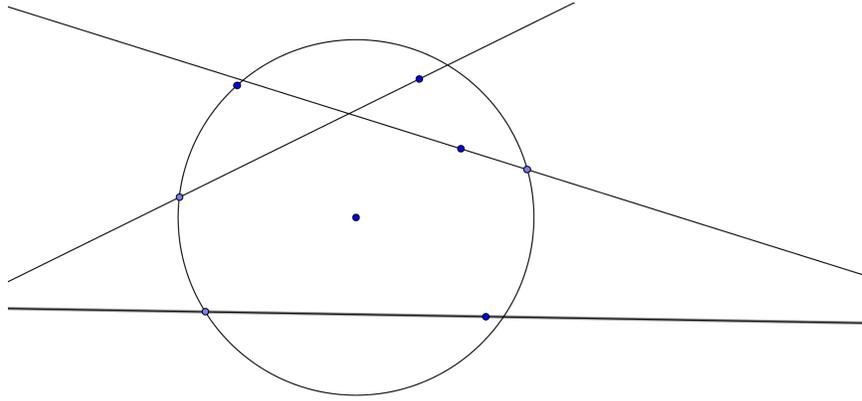For example, Klein disk satisfies the hyperbolic postulate.



FIGURE 3. The Klein Disk

## 2. **Hyperbolic Geometry**

These five postulates are what governs Euclids geometric world and allowed him to formulate the extensive proofs that are what we know today (Joyce).

For years mathematicians used Euclids theorems and postulates in an effort to prove that the fifth postulate was either a theorem, meaning that it was based on the other four postulates, or they attempted to modify it. Finally, in the 19th century, two mathematicians published two separate descriptions of a geometry that satisfied all but the fifth of Euclids postulates. The two mathematicians were Euginio Beltrami and Felix Klein and together they developed the first complete model of hyperbolic geometry. This description is now what we know as hyperbolic geometry (Taimina). In Hyperbolic Geometry, the first four postulates are the same as Euclids geometry. However, the fifth postulate is changed to Hyperbolic Parallel Postulate:

(1) Given a point not on a line, there exists more than one line passing through the point and not meeting the given line, no matter how far it is extended.

By changing this last axiom, it allowed for a whole new geometry to be formed with a new set of theorems and definitions. This alone is what allowed for the core of Euclidean Geometry to be changed and made way for many other types of geometries.

Having set the foundation for Hyperbolic Geometry, this next section will look into three different models, providing a general description and foundation for each.

## 3. **Hyperbolic Models**

From now on, we assume the existence of a model for Euclian Geometry. Within the euclidian model we will construct a hyperbobil model.

3.1. **Poincare Disk Model.** The first model is the Poincar Disk model. Let $\delta$ be a fixed circle in the euclidian plane.

(1) A point in the Poincare Disk Model is any point inside the fixed circle $\delta$.
(2) A line in the Poincare Disk Model can be formed by all the inside points of an open diameter of $\delta$ or can be formed by all the points inside of a circle $\beta$ orthogonal to the given circle $\delta$.
(3) A half-plane in the Poincare Disk Model is given by the intersection points between the given circle $\delta$ and the euclidian half-plane determined by a line, if this a line passing through the center of the circle $\delta$, or the intersection points between of the interior and exterior of $\beta$ and the interior of $\delta$.
(4) For the Poincar Disk Model, in order to define distance, the measure of an angle must first be defined. Since the rays of Hyperbolic Geometry are curved, the measurement of the angle between them is the measure of the Euclidean angle between the rays, as shown in Figure 4. Angles in the Poincare Disk Model. The

measurement of $\angle ABC$ is the measurement of the Euclidean angle $\angle A'BC'$, using the lines $A'B$ and $C'B$ which are tangent to the arcs at point $B$. (Venema).
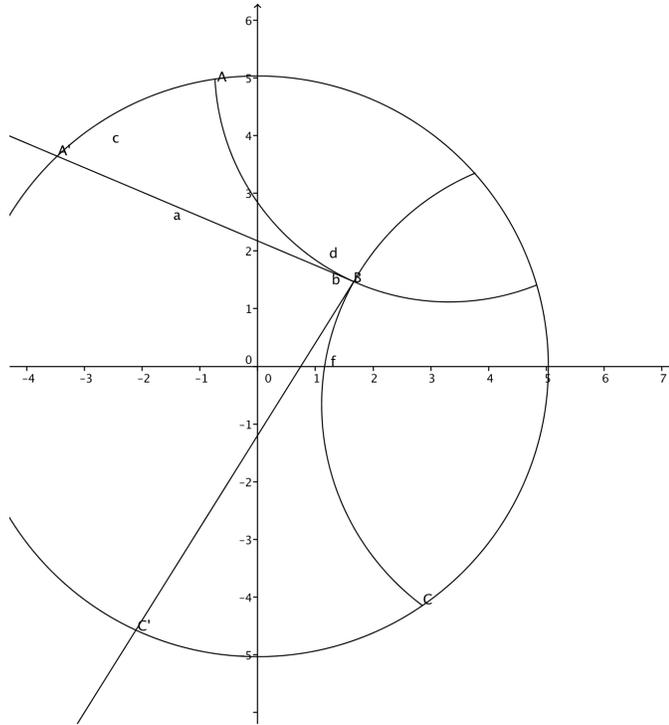


FIGURE 4. Angles in the Poincare Disk Models

(5) The distance from $A$ to $B$ on the Poincare Disk Model is defined by the cross-ration

$$[AB, PQ] = \frac{(AP)(BQ)}{(AQ)(BP)},$$

where $P$ and $Q$ are the end points of the diameter containing $A$ and $B$ or they are the intersection points between the unique circle containing $A$ and $B$ that is orthogonal to the fixed circle and the fixed circle.

Note: the points $P$ and $Q$ are not points of the Poincare Disk Model, but points on the given circle, they may be view as $\infty$ for numbers.

3.2. **The Beltrami-Klein Disk Model.** The next model is the Beltrami-Klein, or sometimes just called the Klein Model.

(1) A point in this model is an euclidian point inside of a given circle $\delta$.
(2) A line in this model is the portion of the euclidian line that lies inside $\delta$.
(3) Unlike the Poincar Disk model, we do not define or interpret distance and angle measures to begin. This model instead uses a transformation $h$, which maps the Klein Disk Model to the Poincar Disk Model so that lines are mapped onto lines. The distance, or angle measure, is then found by viewing their partner lines in the Pioncar Disk under $h$ (Venema).
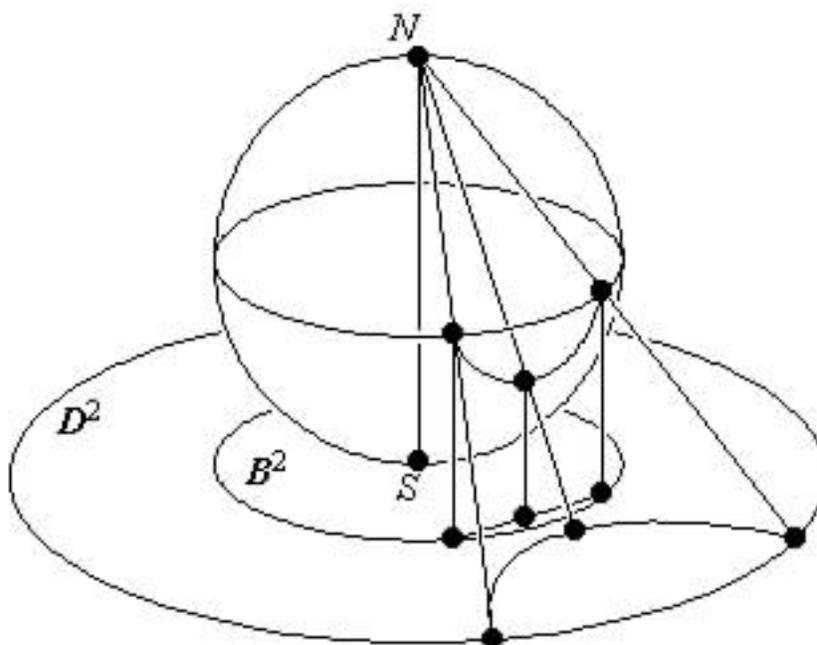


FIGURE 5. The Isomorphism h

Figure 5. is a diagram of the use of the transformation $h$, which is how points and lines are sent from the Klein Disk Model to the Poincar Disk Model.

This transformation begins by taking a sphere which is tangent to the Euclidean plane at point $S$, with $B^2$ being a circle directly under the equator of the sphere(the projection of the equator onto the plane).

The points on the Klein Disk are the points inside $B^2$. The first half of the $h$ transformation is from the Klein Disk to the lower hemisphere of the sphere, is applied by taking a point in $B^2$ and simply moving up until it hits the sphere.

This transformation named $f$ can be performed either way and is specific to the lower half of the sphere only.

The second half of the transformation, named $g$, can be used for any point of the sphere except $N$. It is called the stereographic projection, and sent any point from the sphere to a point on the plane by intersection of the line passing through $N$ and any point of the sphere with the plane.

If you compose the two transformation the result is $h = f \circ g$. This transformation is invertible:

if you choose a point in $D^2$ and then connecting it by a line to the the north pole of the sphere and then take the point where the line intersects the sphere and simply drop it down until it hits a point of the Euclidean plane.

This transformation can be used either way making it very versatile (Venema).

The main reason that the Klein model is used, is that it is easy to define the points and lines.

However, this model is not conformal. This means that the distance and angle measurements are not always accurate, and for this reason explicit formulas for distance and angle measurements are not taken from the Klein model. Rather they are defined indirectly using $h$ (Venema).

3.3. **The Poincare Half-Plane Model.** Another model is the Poincar half-plane model, and is illustrated in Figure 6. We begin by choosing a line which lies in the Euclidean plane and one of the two half planes it divides.
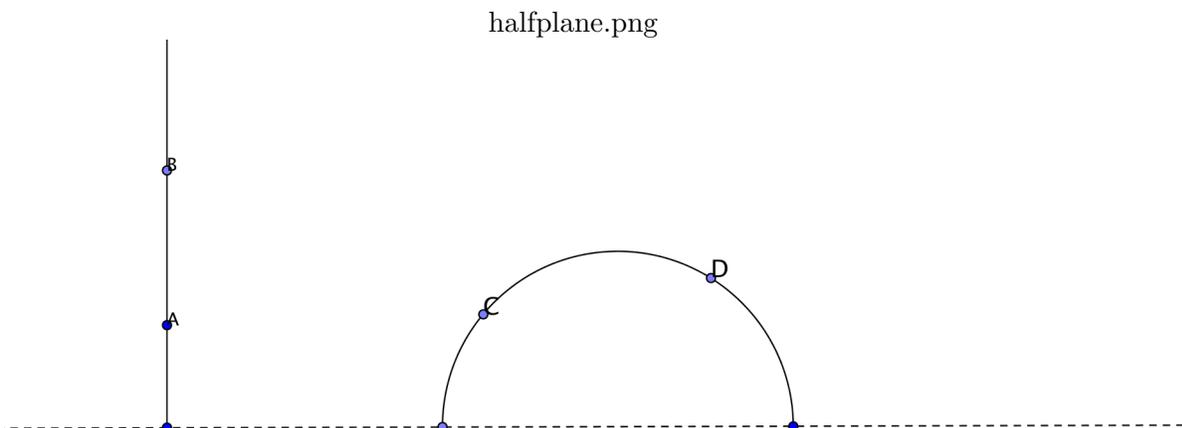
halfplane.png

FIGURE 6. The Poincare Half-Plane Model

(1) For this model a point is simply one that is a Euclidean point in that half-plane.
(2) For a line there are two definitions. The first being; that the half of the line which is included in the half-plane chosen and is also perpendicular to $\lambda$, is a line. The second definition of the line includes all of the points of a Euclidean circle that is perpendicular to $\lambda$, and lie in the half-plane (Venema).
(3) The isomorphism for this model is very similar to the Beltrami-Klein model. This model works by taking a sphere and having it be tangent to the plane, which is a line in a vertical Euclidian plane. The isomorphism of this model is used by taking a point in the Euclidean plane and connecting it to the pole of the sphere creating a line. Again, the point where the line intersects the sphere is the point that correlates to the sphere. This can also be used in either direction making it very versatile (Venema).

3.4. **The hyperbolic Model.** Let take the hyperboloid of two sheets: $z^2 = 1 + x^2 + y^2$ in $R^3$, which is obtained by revolving the hyperbola $z^2 - x^2 = 1$ around the $z$-axis.
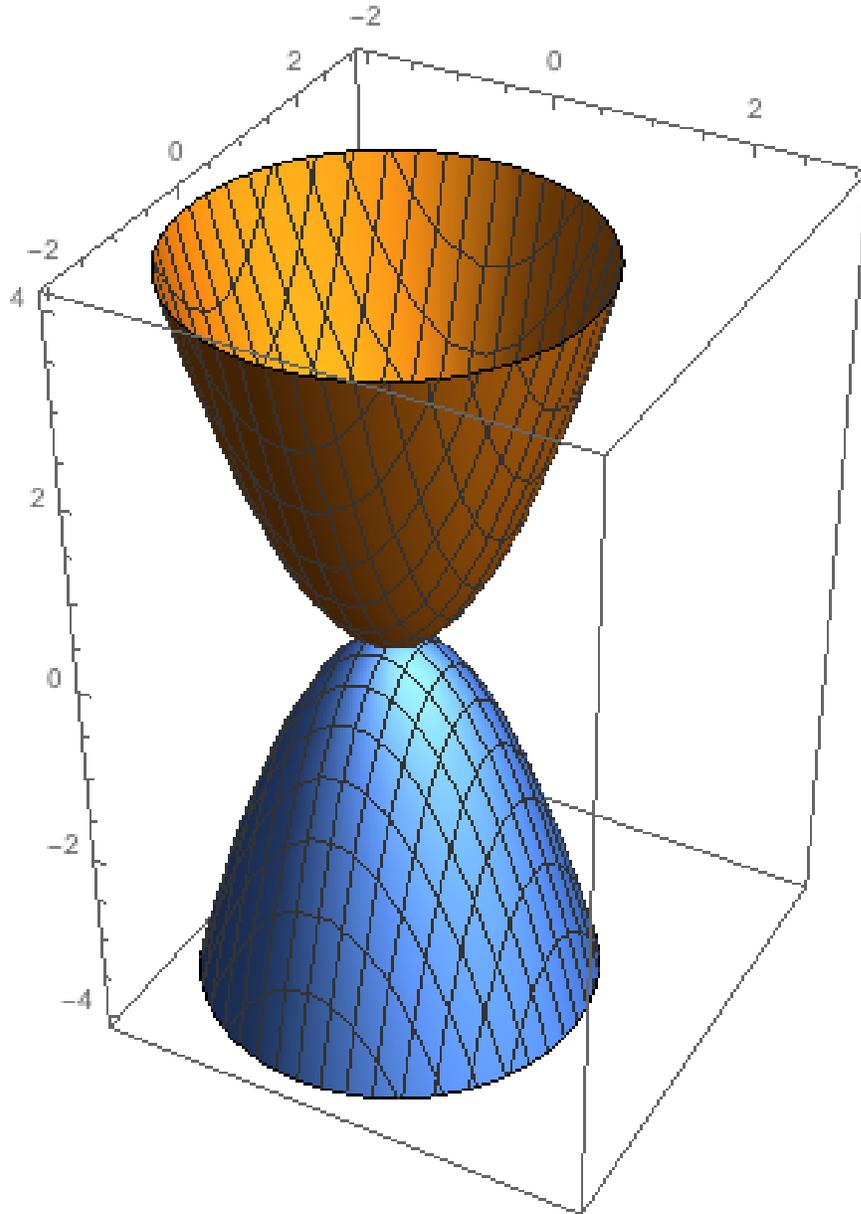


FIGURE 7. Hyperboloid

(1) A point in this model is a euclidian point in $R^3$ that lies on the hyperboloid such that $z > 0$
(2) A line in this model is the intersection of a plane through origin with the upper sheet of the hyperboloid.
(3) To define the distance we will use an isomorphism, as for the previous two models. The isomorphism is defined by taking points and lines in this models and transforms them into points and lines in one of the previous model.
The circle  that lies in the plane $z = 1$ and has the center $(0, 0, 1)$ is the Klein Disk. The radial projection from the origin is the isomorphism. A plane $\alpha$ intersects the paraboloid along a line $l'$ and the Klein Disk along the line $l$. The isomorphism between the Klein Disk and the hyperboloid takes $l$ into $l'$.

## 4. **Conclusion**

In conclusion, while Euclidean geometry is the most common and well known of any type of geometry, a simple modification to its postulates can create a whole new set of geometric properties. These different geometries are infinitely useful and have answered and proved many different theories.For example the relativity theory is constructed in a hyperbolic geometry. It gives mathematicians the options and insight they need to continually progress in the mathematical world.

### References

[1] Artmann, Benno *Euclidean Geometry*, Encyclopedia Britannica Online. Encyclopedia Britannica, 17 Nov. 2014. Web. 18 Apr. 2015. ¡http://www.britannica.com/EBchecked/topic/194901/Euclidean-geometry¿.
[2] Cannon, James, William Floyd, Richard Kenyon, and Walter Parry *Hyperbolic Geometry*, Flavors of Geometry 31 (1997). Web. 12 Apr. 2015. ¡http://library.msri.org/books/Book31/files/cannon.pdf¿.
[3] Joyce, David*Euclid's Elements, Book I* Web. 18 Apr. 2015. ¡http://aleph0.clarku.edu/ djoyce/elements/bookI/bookI.html¿.
[4] Taimina, Daina.*Hyperbolic Geometry*Encyclopedia Britannica Online. Encyclopedia Britannica, 23 Mar. 2014. Web. 18 Apr. 2015. ¡http://www.britannica.com/EBchecked/topic/417456/non-Euclidean-geometry/235579/Hyperbolic-geometry¿.
[5] Venema, Gerard*Foundations of Geometry* 2nd ed. Boston: Pearson, 2012. Print.

Department of Mathematics, Medaille College, Buffalo, NY 14214
*E-mail address*: vm72@medaille.edu